Systematic skew trading

Guanlin Li

risk reversal, *i.e.,* long (short) put and short (long) call, is a characteristic position to trade skew dynamics. For a delta-hedged and spot gamma neutral r.r position, there are three major risk attributions, skew decay (via theta exposure), volatility surface remark (via vanna exposure) and skew rotation dynamics (via vega exposure). To make notional gamma neutral, we construct a ratio r.r, long K_p strike put and short λ units of call at strike K_c,

$$r.r = P(F, \sigma_{K_p}; K_p) - \lambda C(F, \sigma_{K_c}; K_c),$$

where K_p and K_c are r.r out-the-money call- and put-strikes, σ_{K_p} and σ_{K_c} are implied volatilities of call and put, respectively. Recall gamma-theta relationship from black model equation (set risk-free interest rate $r = 0$),

$$\Theta + \frac{1}{2}\sigma^2 F^2 \Gamma = 0.$$

The profit and loss analysis of above r.r position (delta hedged) can be worked out as following,

$$d(r.r) = \underbrace{\frac{1}{2}\Gamma_{K_p} F^2 (\sigma_{K_c}^2 - \sigma_{K_p}^2) dt}_{\text{skew theta carry pl}} + \underbrace{(v_{K_p} d\sigma_{K_p} - \lambda v_{K_c} d\sigma_{K_c})}_{\text{skew dynamics pl}} + \underbrace{(w_{K_p} d\sigma_{K_p} dF - \lambda w_{K_c} d\sigma_{K_c} dF)}_{\text{realized skew pl}}, \quad (1)$$

where $v_{(.)}$, $w_{(.)}$ are vega and vanna exposures, respectively. Note that, we let

$$\lambda = \Gamma_{K_p} / \Gamma_{K_c},$$

in doing so, r.r is a spot Gamma neutral position. Long/short r.r (or reverse r.r) depends on the expected profit of r.r, namely, $\alpha = \mathbf{E}[d(r.r)]$, the size of skew alpha is quantified by the notional cashVanna of target r.r position, the expected skew pl per notional cashVanna will be our final indicator to make trade decision.

The implied volatility values, σ_{K_c} and σ_{K_p}, can be parameterized by an at-the-money volatility and a skew parameter. Denote $\sigma_0 = \sigma(K_F)$ as the volatility of the strike at spot price, *i.e.*, $K_F = F$, for a put-call r.r spread, K_F is usually the middle strike of K_p and K_c. We compute the dimensionless *implied skew* s,

$$s = \frac{\sigma_{K_c} - \sigma_{K_p}}{(K_c - K_p)/F}. \quad (2)$$

There are only two volatility values in r.r, $(\sigma_{K_c}, \sigma_{K_p})$, the volatility pair can be represented in a level-skew coordinate with no cost. In doing so, we can rewrite put- and call-strike volatility values (using linear expansion),

$$\sigma_{K_{(.)}} = \sigma_0 + \frac{K_{(.)} - F}{F} s.$$

However, a volatility smile with large atm-convexity could induce a non-negligible offset in above equation. The volatility risk can be decomposed into two factors, surface level shift and skew rotation,

$$d\sigma_{K_{(.)}} = d\sigma_0 + \frac{K_{(.)} - F}{F} ds. \quad (3)$$

We should be cautioned about $d\sigma_0$. In our system, we use skew-swimming ratio (ssr) to modulate the SVI forward input between a reference strike (e.g. ssr = 0, shadow delta is raw delta) and the spot price (e.g. ssr = 1, sticky-delta shadow delta). When $ssr = 0$, σ_0 is same as the SVI parameter, atm-volatility. If ssr = 1, then the SVI parameter, atm-volatility, is the atm-forward volatility, the movement of surface can be computed as

$$d\sigma_0 = d\sigma_{ATM} - \frac{dF}{F} s, \quad (4)$$

where σ_{ATM} is the atm-volatility. Note that *sticky-to-strike* implies $d\sigma_{ATM} = \frac{dF}{F}s$ thus $d\sigma_0 = 0$. Using skew approximation form, see Eq. 3, we can rewrite the skew dynamics pl attribution (see Eq, 1),

$$\begin{aligned} \textit{skew dynamics pl} &= (v_{K_p} d\sigma_{K_p} - \lambda v_{K_c} d\sigma_{K_c}) \\ &= v_{K_p}\left(d\sigma_{K_p} - \frac{\lambda v_{K_c}}{v_{K_p}} d\sigma_{K_c}\right) \\ &\approx v_{K_p}\left(x_{K_p} - \frac{\lambda v_{K_c}}{v_{K_p}} x_{K_c}\right) ds, \end{aligned} \tag{5}$$

where $x_K = (K - F)/F$ is the strike moneyness, the approximation is hold when $1 - \frac{\lambda v_{K_c}}{v_{K_p}}$ is small, i.e., $\frac{\lambda v_{K_c}}{v_{K_p}} \approx 1$. The realized skew pl (vanna pl) can be rewrite as

$$\begin{aligned} \textit{realized skew pl} &= (w_{K_p} d\sigma_{K_p} dF - \lambda w_{K_c} d\sigma_{K_c} dF) \\ &= w_{K_p}\left(d\sigma_{K_p} - \frac{\lambda w_{K_c}}{w_{K_p}} d\sigma_{K_c}\right) dF \\ &\approx w_{K_p}\left(1 - \frac{\lambda w_{K_c}}{w_{K_p}}\right) d\sigma_0 dF, \end{aligned} \tag{6}$$

the approximation is made when $\frac{\lambda w_{K_c}}{w_{K_p}} \approx -1$, *note that vanna pl is resulted from spot and surface level movement (NOT surface skew rotation).* The statistical average of realized skew risk, $d\sigma_0 ds$, can be computed as

$$\begin{aligned} \mathbf{E}[d\sigma_0 ds] &= F\operatorname{cov}(d\sigma_0, dF/F) \\ &= F\operatorname{var}(dF/F)\frac{\operatorname{cov}(d\sigma_0, dF/F)}{\operatorname{var}(dF/F)} \\ &= F(\sigma^2_{realized} dt)\hat{\beta}, \end{aligned} \tag{7}$$

where $\hat{\beta}$ is the fitted regression slope between underlying movement and volatility surface shift movement, $\sigma^2_{realized} dt$ is the total variance in the carry horizon. The expected pl of realized skew is

$$\textit{expected realized skew pl} = w^*_{K_p}\left(1 - \frac{\lambda w_{K_c}}{w_{K_p}}\right)(\sigma^2_{realized} dt)\hat{\beta}, \tag{8}$$

where $w^*_{K_p} = w_{K_p} F$ is cash vanna. Together, we calculate the expected pl of a r.r position,

$$\begin{aligned} \mathbf{E}[d(r.r)] &= \frac{1}{2}\Gamma_{K_p} F^2(\sigma^2_{K_c} - \sigma^2_{K_p})dt + v_{K_p}\left(x_{K_p} - \frac{\lambda v_{K_c}}{v_{K_p}} x_{K_c}\right)\mathbf{E}[ds] + w^*_{K_p}\left(1 - \frac{\lambda w_{K_c}}{w_{K_p}}\right)(\sigma^2_{realized} dt)\hat{\beta} \\ &= \underbrace{\left(\frac{1}{2}\Gamma_{K_p} F^2(\sigma^2_{K_c} - \sigma^2_{K_p}) + w^*_{K_p}(1 - \lambda_w)\sigma^2_{realized}\hat{\beta}\right) dt}_{\text{expected skew carry pl}} + \underbrace{v_{K_p}\left(x_{K_p} - \lambda_v x_{K_c}\right)\mathbf{E}[ds]}_{\text{skew dynamics pl}}, \end{aligned} \tag{9}$$

where $\lambda_v = \frac{\lambda v_{K_c}}{v_{K_p}} > 0$ and $\lambda_w = \frac{\lambda w_{K_c}}{w_{K_p}} < 0$.

Remark 1. *The skew dynamics pl can be rewritten in terms of volatility spread,*

$$\textit{expected skew dynamics pl} \approx v_{K_P}\mathbf{E}[\textit{vol spread change}],$$

where vol spread change is the change of $\sigma_{K_P} - \sigma_{K_C}$. Noting that σ_{K_P} and σ_{K_C} are (25)/25 delta strike volatility values, the vol spread between two fixed delta-space strikes is time-invariant.

Sticky-delta dynamics. Suppose volatility surface dynamics follows sticky-to-delta dynamics, atm-skew is positive. We hold a gamma-neutral position of risk reversal, sell call and long put (i.e., sell skew), we will collect skew theta premium and realize a positive volatility surface remark pl after underlying move (net skew carry is positive), however, the skew dynamics risk might be large. For example, if current atm-skew is at the lower quantile of historical distribution, we will suffer a huge loss for the upside movement in skew (as we short skew). The most promising trade would be a 'win-win' trade, trade with positive carry and positive skew edge.

A time-invariant skew measurement. The implied skew metric, see Eq. 2, is a dimensionless quantity, yet it's not time-invariant. Suppose K_c and K_p are 25/(25) delta strikes, the volatility spread between two fixed moneyness points should be time-invariant, however, the strike difference, $K_c - K_p$, will get smaller when it comes to maturity. Therefore, the variance of implied skew s gets larger near expiration. We normalize our implied skew metric as following,

$$s' = s * (\sigma_{ATM}\sqrt{t}). \tag{10}$$

The quantitative reasoning of above skew normalization form is worked out in another quantitative analysis paper, long story short, volatility skew is time-invariant in delta space $\sim \log(K/F)/(\sigma\sqrt{t})$.

Mean reversion of normalized skew. We assume the normalized skew follows a mean reverting dynamics, e.g. the time series of normalized skew can be fitted to a first order auto-regression model. Using some signal filter technique, the instant stationary level can be carried out, we denote it as s'_{theo}. Hence, the instant skew edge based on a reverting model is,

$$\begin{aligned} \mathbf{E}[ds'] = s'_{theo} - s' \quad \Rightarrow \quad \mathbf{E}[ds] &= \mathbf{E}[ds']/(\sigma_{ATM}\sqrt{t}) \\ &= \frac{s'_{theo} - s'}{\sigma_{ATM}\sqrt{t}}. \end{aligned} \tag{11}$$

The above skew edge will be substituted into Eq. 9 to calculate the edge of holding a r.r position. We propose a risk-adjusted edge measurement, *expected pl of a r.r position per notional cash vanna.* The 'alpha' of entering a r.r position is,

$$\begin{aligned} \alpha_{rr} &= \frac{\mathbf{E}[d(r.r)]}{w^*_{rr}} \\ &= \frac{1}{w^*_{rr}}\left(\left(\frac{1}{2}\Gamma_{K_p}F^2(\sigma^2_{K_c} - \sigma^2_{K_p}) + w^*_{K_p}\left(1-\lambda_w\right)\sigma^2_{realized}\hat{\beta}\right)dt + v_{K_p}\left(x_{K_p} - \lambda_v x_{K_c}\right)\frac{s'_{theo} - s'}{\sigma_{ATM}\sqrt{t}}\right), \end{aligned} \tag{12}$$

where $w^*_{rr} = (w_{K_p} - \lambda w_{K_c})F$ is the notional cash vanna of r.r. We use mean-variance framework to manage our trading position (towards *Sharpe maximization*), the real-time position Q can be computed as

$$Q \sim \frac{1}{c} * \frac{\alpha_{rr}}{\mathrm{var}(r.r/w^*_{rr})}, \tag{13}$$

where c is the risk aversion factor. Noting that above position management is written in a 'qualitative' form, in the real world, we should generate trade position with at least α_{min} edge and cap the holdings at some maximum level.

Vanilla trading strategy. For commodity options (with day and night trading hours), we trade at day and night openings, e.g. 9:10 and 21:10. To calibrate skew carry and skew statistical model (mean-reversion), we must use low-frequency data, e.g. using daily close-to-close data to fit realized skew correlation, $\hat{\beta}$. Based on our sampling and trading frequency, we could calibrate parameters using last one- or two-week data. We constantly keep our r.r position at 25/(25) delta, so we must rolling the call and put in the holdings, for example, if spot rallies 10%, then the 25 delta call becomes at-the-money call, in such case, we will roll over the put and call in the r.r. We set a roll-over frame, say, $[15, 35]$ delta range, if one leg in r.r. falling out of this range, we automatically roll over to the nearest 25-delta strikes. In addition, we need to constantly re-balance delta and gamma exposures to make the position delta- and gamma-neutral.