The *indifference* price

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This technical note expands the framework for determining indifference prices [AS08], a heuristic method used to optimize the trading behavior of a market maker (e.g., pricing and risk management), into the options market-making system. Additionally, we integrate margin costs into the utility function. First, we review the basic model presented in [AS08]. The mid-price of the asset (e.g. a stock), S, follows the geometric Brownian motion,

$$\frac{dS(t)}{S} = \sigma dW(t), \tag{1}$$

where σ is the constant volatility and $W(t)$ is a standard Brownian motion, with initial state $S(0) = s$. We consider a market maker holds inventory q, cash x, the mean-variance utility function, evaluated at time t, can be written as,

$$\begin{aligned} V_t(x, s, q) &= \mathbf{E}\left[\overbrace{(x + qS(t))}^{\text{terminal wealth}} - \frac{\gamma}{2} \overbrace{(qS(t) - qs)^2}^{\text{variance}} \right] \\ &= x + qs - \frac{\gamma}{2} q^2 s^2 \overbrace{(e^{\sigma^2 t} - 1)}^{\sim\ \sigma^2 t}, \end{aligned} \tag{2}$$

where x is the initial wealth in dollars, we can set it to zero. The *reservation bid price* r^b is solved by $V_t(x - r^b, s, q+1) = V_t(x, s, q)$, the *reservation ask price* r^a is solved by $V_t(x + r^a, s, q-1) = V_t(x, s, q)$. This yields reservation prices of the form,

$$\begin{aligned} r_t^a(s, q) &= s - \gamma s^2 \sigma^2 t q + \frac{1}{2} \gamma s^2 \sigma^2 t, \\ r_t^b(s, q) &= s - \gamma s^2 \sigma^2 t q - \frac{1}{2} \gamma s^2 \sigma^2 t. \end{aligned} \tag{3}$$

Note that $s^2 \sigma^2 t$ is the spot dollar variance, the reservation price is the mid-price adjusted by risk management shift (linear in q) and required spread (linear in variance).

We extend above approach to the options market-making system. We use black-scholes model, where option price is the function of spot price S and implied volatility σ. In general, we define option price, $f(Z_1, Z_2, ..., Z_m; k, \tau)$, as a pricing function of multiple market variables, $Z_1, ..., Z_m$, with strike k and time-to-maturity τ. Suppose the terminal time t is small, the option price can be written in it's expansion form,

$$\begin{aligned} f(Z_1(t), ..., Z_m(t)) \sim\ & f(Z_1(0), ..., Z_m(0)) + \sum_i (Z_i(t) - Z_i(0)) \frac{\partial f}{\partial Z_i}(Z_1(0), ..., Z_m(0)), \\ \text{in short,}\ & f(t) \sim f(0) + \sum_i \delta Z_i \frac{\partial f(0)}{\partial Z_i}. \end{aligned} \tag{4}$$

Furthermore, we take into account the margin cost of trading options (i.e., the financing cost). The margin might depend on the long/short position, for example, there is no margin for buying options (only need to pay option premiums), however, the margin of selling options is large. Thus, the `optionMargin` is a digital function of position. The margin cost of holding an option with inventory q is,

$$\overbrace{\left[-rt * \text{optionMargin} * \mathbb{1}_{\{q<0\}}\right]}^{\text{cost form, } c} q. \tag{5}$$

The cost form, c, is a indicator function, which can be approximated by a smooth sigmoid function. The discontinuity issue (around $q = 0$) is omitted in forming the utility function (see Eq. 6) by introducing an assumption on position variation range. We consider a portfolio of N options, the position vector is denoted by $\mathbf{q} \in \mathbb{R}^N$. Note that the bold symbols correspond to vector forms. The utility function has form,

$$V_t(x, \mathbf{f}(0), \mathbf{q}) = \mathbf{E}\left[\overbrace{(x + \mathbf{q}^T\mathbf{f}(t))}^{\text{terminal wealth}} - \frac{\gamma}{2}\overbrace{\left(\mathbf{q}^T\mathbf{f}(t) - \mathbf{q}^T\mathbf{f}(0)\right)^2}^{\text{variance}} - \phi \overbrace{\mathbf{q}^T\mathbf{c}(0)}^{\text{margin cost}}\right], \tag{6}$$

where the margin cost vector $\mathbf{c}(0) \in \mathbb{R}^N$ is a constant vector evaluated at initial state given position $\mathbf{q}$ and market state $\mathbf{Z}(0) \in \mathbb{R}^m$, the element of $\mathbf{c}(0)$ has form as Eq. 5. Substituting expansion form, Eq. 4, into the above utility function, we have closed form,

$$\begin{aligned} V_t(x, \mathbf{f}(0), \mathbf{q}) &= x + \mathbf{q}^T\mathbf{f}(0) - \frac{\gamma}{2}\mathbf{q}^T\Omega\mathbf{q} - \phi\mathbf{q}^T\mathbf{c}(0), \\ \text{where } \Omega &= \mathbf{E}\left[\sum_i\sum_j \delta Z_i \delta Z_j \mathbf{g}_i\mathbf{g}_j^T\right], \end{aligned} \tag{7}$$

the risk sensitivity shape with respect to risk factor Z_i, $\partial\mathbf{f}(0)/\partial Z_i \in \mathbb{R}^N$, is denoted by $\mathbf{g}_i$. We assume there is no correlation between any pair of market risk factors, $\delta Z_i \perp\!\!\!\perp \delta Z_j$ for any $i \neq j$, the covariance structure is simplified to

$$\Omega = \sum_i \lambda_i \mathbf{g}_i\mathbf{g}_i^T, \ \Omega \in \mathbb{R}^{N\times N} \tag{8}$$

where $\lambda_i = \mathbf{E}[\delta Z_i^2]$ is the variance of risk factor δZ_i, it depends on the trading horizon (t), the variance gets larger for longer time. We find the indifference price (or the reservation price), $\mathbf{f}_{quote}$, by solving the equation $V_t(x - \mathbf{f}_{quote}\mathbf{h}^T, \mathbf{f}(0), \mathbf{q}+\mathbf{h}) = V_t(x, \mathbf{f}(0), \mathbf{q})$, where $\mathbf{h}$ is a position incremental vector. Note that we do not differentiate between long and short trades, the indifference utility applies to any trade vector. The position incremental vector should be small, so that the margin cost remains same for all contracts after adding a small position change $\mathbf{h}$, e.g. $\text{sign}(\mathbf{q}) = \text{sign}(\mathbf{q}+\mathbf{h})$. We solve for $\mathbf{f}_{quote}$ as follows,

$$\begin{aligned} x - \mathbf{f}_{quote}\mathbf{h}^T + (\mathbf{q}+\mathbf{h})^T\mathbf{f}(0) - \frac{\gamma}{2}(\mathbf{q}+\mathbf{h})^T\Omega(\mathbf{q}+\mathbf{h}) - \phi(\mathbf{q}+\mathbf{h})^T\mathbf{c}(0) &= x + \mathbf{q}^T\mathbf{f}(0) - \frac{\gamma}{2}\mathbf{q}^T\Omega\mathbf{q} - \phi\mathbf{q}^T\mathbf{c}(0), \\ -\mathbf{f}_{quote}\mathbf{h}^T + \mathbf{h}^T\mathbf{f}(0) - \frac{\gamma}{2}\left(2\mathbf{h}^T\Omega\mathbf{q} + \mathbf{h}^T\Omega\mathbf{h}\right) - \phi\mathbf{h}^T\mathbf{c}(0) &= 0, \\ -\mathbf{h}^T\left(\gamma\Omega\mathbf{q} + \frac{\gamma}{2}\Omega\mathbf{h} + \phi\mathbf{c}(0)\right) &= \mathbf{h}^T(\mathbf{f}_{quote} - \mathbf{f}(0)). \end{aligned} \tag{9}$$

The above equation holds for any $\mathbf{h}$, we have

$$\mathbf{f}_{quote} = \mathbf{f}(0) + \overbrace{(-\gamma\Omega\mathbf{q})}^{\text{position lean}} + \overbrace{\left(-\frac{\gamma}{2}\Omega\mathbf{h}\right)}^{\text{risk spread}} + \overbrace{(-\phi\mathbf{c}(0))}^{\text{margin adjust}}. \tag{10}$$

Note that

$$\Omega\mathbf{q} = \sum_i \lambda_i \overbrace{(\mathbf{q}^T\mathbf{g}_i)}^{\text{factor } i\text{'s risk position}} \mathbf{g}_i, \tag{11}$$

this is equivalent to the risk management model in our market making system, we adjust the factor value Z_i proportional to it's risk position. To see this, we write the adjustment on factor Z_i in terms of it's risk position, $\Delta Z_i = \beta_i\mathbf{q}^T\mathbf{g}_i$, where β_i is a control parameter. Next, we calculate the option price change vector, $\mathbf{f}(\mathbf{Z} + \Delta Z_i\mathbf{1}_i) - \mathbf{f}(\mathbf{Z}) \sim \Delta Z_i\mathbf{g}_i$, where $\mathbf{1}_i$ is the zero-one coded vector with value one at index i. This form recovers the *penalty* function in our system. For example, if we sell a million cash delta, we would increase the underlying price by one tick.

The second term, *risk spread*, is linear in trade size vector $\mathbf{h}$, we require more spread (per lot) for large trade, *i.e.,* size-edge relationship. The third term, $\phi\mathbf{c}(0)$, accounts for the margin cost adjustment, it's not depending on position.

References

[AS08] Marco Avellaneda and Sasha Stoikov. High-frequency trading in a limit order book. *Quantitative Finance*, 8(3):217–224, 2008.